Unified Engineering Fall 2007
Signals & Systems Problems

Assigned: Friday, October 19
Due: Friday, October 26
Note: Please hand in problems on separate sheets of paper

Problem 1: Look-back to Lecture 3 (15 points)

In Systems Labs 2–6, you used the following velocity rate equation:

$$\dot{V} \quad = \quad -g - \frac{1}{2}\rho V^2 \frac{C_D A}{m} + \frac{T}{m}. \tag{1}$$

To simplify the analysis, we will assume that the rocket is always climbing, so V is positive. We will also assume that the rocket mass m is fixed. In the above equation, we have the following quantities, which are taken as constant for our purposes: g, the gravitational acceleration; C_D, the drag coefficient; A, the reference area; ρ, the air density. The *input* to our system is the applied propulsive thrust $T(t)$, while the *output* of interest is the rocket velocity $V(t)$.

(a) Assume that the rocket is climbing at a steady speed V_0, under the action of a constant thrust T_0. Determine this V_0 as a function of the rocket parameters.

Note: This V_0 is <u>not</u> one of the Initial Conditions you used to integrate the ODEs in Labs 2–6. Those ICs are not relevant in this analysis. We will also assign $t = 0$ to some convenient point along the trajectory, not to the initial time at launch.

(b) Now consider some perturbations $T'(t)$ and $V'(t)$ away from the steady state:

$$\begin{aligned} T(t) &= T_0 + T'(t), \\ V(t) &= V_0 + V'(t). \end{aligned}$$

Derive a linearized model (linear ODE) for the rocket that relates a small perturbation in thrust, $T'(t)$, to a perturbation in velocity, $V'(t)$. You will need to linearize equation (1) about the steady-state T_0, V_0.

(c) Analytically solve the resulting linear ODE to determine an expression for the output $V'(t)$, when the input is a unit step function:

$$T'(t) = \begin{cases} 0\,\mathrm{N} & , \quad t < 0, \\ 1\,\mathrm{N} & , \quad t \geq 0. \end{cases}$$

Note that the initial condition for your ODE is $V'(0) = 0$.

(d) For numerical evaluation and plotting, use the following parameters:

$$\begin{aligned} g &= 9.81\,\mathrm{m/s^2} \\ m &= 0.12\,\mathrm{kg} \\ A &= 0.001\,\mathrm{m^2} \\ \rho &= 1.22\,\mathrm{kg/m^3} \\ C_D &= 0.4 \\ T_0 &= 2\mathrm{N} \end{aligned}$$

Compute the corresponding V_0. Plot the step response $V'(t)$, over a time interval which clearly shows the behavior. Explain why your results are/are not physically reasonable. What are the effects of the assumptions we have made in deriving the linearized model?

(e) Now consider the smaller thrust step change

$$T'(t) = \begin{cases} 0\,\mathrm{N} & , \quad t < 0 \\ 0.1\,\mathrm{N} & , \quad t \geq 0 \end{cases}$$

Determine the resulting $V'(t)$. You should be able to do this by inspection of your result from (c). Explain why this $V'(t)$ is/is not physically reasonable.

Problem 2: Look-ahead to Lecture 4 (15 points)

Reading: ow 2.0, 2.1, 2.3.8.
(Note: don't wait until Wednesday's lecture to start this one!)

In Problem 1(c), you computed the step response of the linearized rocket model, that is, you computed the perturbation in velocity of the rocket in response to a unit step change in thrust. We will denote the unit step response as $V_s(t)$.

We will see in Lecture 4 that Duhamel's superposition integral tells us that the response $y(t)$ of a system to a general input $u(t)$ can be determined from the step response $V_s(t)$ as

$$y(t) \quad = \quad V_s(t)u(0) + \int_0^t V_s(t-\tau)\frac{du(\tau)}{d\tau}\, d\tau.$$

Here, the input is the perturbation in thrust and the output is the perturbation in velocity, i.e. $u(t) = T'(t)$ and $y(t) = V'(t)$ for our rocket example.

(a) The perturbation in thrust is applied as a step input, but in practice it takes some time for the thrust force to be generated (through the physical thrusting mechanism). In reality, our thrust force has the form

$$T'(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t/t_s}, & t > 0, \end{cases}$$

where t_s is known as the time constant, which is a measure of how quickly the thrust rises to the desired value.

Use Duhamel's superposition integral to compute the response (velocity) of the linearized rocket system to a thrust input

$$T'(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & t > 0. \end{cases}$$

(b) Use Duhamel's superposition integral to compute the response (velocity) of the linearized rocket system to a thrust input

$$T'(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-5t}, & t > 0. \end{cases}$$

(c) Using the same parameter values as in Problem 1(d), compute $V'(t)$ for your responses from 2(a) and 2(b). Plot your results, and compare them to the step response you obtained in Problem 1(d). Explain why your results make physical sense.

Problem 3: Look-ahead to Lectures 5 and 6 (10 points)

Reading: ow 2.2, 2.3
(Note: definitely don't wait until Thursday's lecture to start this one!)

(a) Consider an LTI system with input $u(t)$ and output $y(t)$ related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} u(\tau - 2)\, d\tau.$$

What is the impulse response $h(t)$ of the system?

(b) Determine the response of the system in (a) when the input $u(t)$ is given by

$$u(t) = \begin{cases} 0, & t < -1 \\ 1, & -1 < t < 2, \\ 0 & t > 2. \end{cases}$$

(c) Associativity of convolution: Prove the equality

$$[u(t) * h(t)] * g(t) = u(t) * [h(t) * g(t)],$$

where $*$ denotes the convolution operator and $h(t)$ is the impulse response.

Hint: show that both sides of the equation equal

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\tau) h(\mu) g(t - \tau - \mu)\, d\tau\, d\mu.$$

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